# Problem Set: The Diamond Overlapping-Generations Model

Advanced Macroeconomics — Dr Lei Pan — Total: 100 Marks

**Instructions.** Answer all questions. Show all mathematical derivations clearly. Answers without derivation receive limited credit. Time is discrete, $t = 0, 1, 2, \ldots$. Individuals live for two periods. Population and technology satisfy

$$L_t = (1+n)L_{t-1}, \qquad A_{t+1} = (1+g)A_t, \qquad D \equiv (1+n)(1+g).$$

There is no capital depreciation. Define capital per unit of effective labour as

$$k_t \equiv \frac{K_t}{A_t L_t}.$$

## Question 1: Competitive Equilibrium and Endogenous Saving [Total: 60 marks]

Consider the Diamond economy in which a young individual born at $t$ has lifetime utility

$$U(c_{1t}, c_{2,t+1}) = u(c_{1t}) + \beta u(c_{2,t+1}), \qquad 0 < \beta < 1,$$

with CRRA period utility

$$u(c) = \frac{c^{1-\theta} - 1}{1-\theta}, \qquad \theta > 0,$$

and $u(c) = \ln c$ when $\theta = 1$. The individual budget constraints are

$$c_{1t} + s_t = A_t w_t, \qquad c_{2,t+1} = (1 + r_{t+1})s_t.$$

Firms operate the CRS technology

$$Y_t = F(K_t, A_t L_t^D) = A_t L_t^D f(k_t), \qquad k_t = \frac{K_t}{A_t L_t^D}.$$

**(a)** Derive the Euler equation for the young individual's saving problem. Then derive the saving function

$$s_t = \sigma(r_{t+1})A_t w_t, \qquad \sigma(r) = \frac{\beta^{1/\theta}}{\beta^{1/\theta} + (1+r)^{1-1/\theta}}.$$

Explain briefly why the Inada condition rules out zero consumption.

**(b)** Derive $\sigma'(r)$ and determine whether the saving rate rises or falls with the interest rate when $\theta < 1$, $\theta = 1$, and $\theta > 1$. Give the economic interpretation in terms of substitution and income effects.

**(c)** Solve the firm's profit-maximisation problem. Derive

$$r_t = f'(k_t), \qquad w_t = f(k_t) - f'(k_t)k_t.$$

Then show why CRS and perfect competition imply zero profits.

**(d)** Use labour-market clearing $L_t^D = L_t$ and capital-market clearing $K_{t+1} = L_t s_t$ to derive the implicit transition equation

$$k_{t+1} = \frac{1}{D}\sigma\big(f'(k_{t+1})\big)\left[f(k_t) - f'(k_t)k_t\right].$$

Explain why this equation fully characterises the equilibrium path of $k_t$.

## Question 2: Log Utility, Steady State, Golden Rule, and Dynamic Inefficiency [Total: 40 marks]

Assume now that $\theta = 1$ and

$$f(k) = k^\alpha, \qquad 0 < \alpha < 1.$$

Hence $\sigma(r) = \beta/(1+\beta)$.

**(a)** Derive the explicit transition equation

$$k_{t+1} = Bk_t^\alpha, \qquad B \equiv \frac{\beta}{1+\beta}\frac{1-\alpha}{D}.$$

Solve for the unique positive steady state $k^*$ and prove global stability. Derive the log-linear convergence coefficient and the half-life of deviations from steady state.

**(b)** For

$$\alpha = \frac{1}{3}, \qquad \beta = 0.96, \qquad n = 0.01, \qquad g = 0.02,$$

compute $D$, $B$, $k^*$, $y^*$, $w^*$, and $r^*$. State the balanced-growth-path growth rates of aggregate output, aggregate capital, aggregate consumption, output per worker, and capital per worker.

**(c)** Derive the Golden Rule capital stock. Starting from the aggregate resource constraint with no depreciation, show that steady-state consumption per unit of effective labour is

$$c(k) = f(k) - (D-1)k.$$

Then derive

$$f'(k_{\text{GR}}) = D - 1, \qquad k_{\text{GR}} = \left(\frac{\alpha}{D-1}\right)^{1/(1-\alpha)}.$$

State the dynamic-efficiency condition.

**(d)** Using the numerical parameters in part (b), compare $k^*$ with $k_{\text{GR}}$ and compare $r^*$ with $D-1$. Is the economy dynamically inefficient? Then derive the general condition for overaccumulation in the log–Cobb–Douglas Diamond economy.